THE SECOND-ORDER SHALLOW-WATER HYDRODYNAMIC
COUPLING COEFFICIENT IN INTERPRETATION OF HF
RADAR SEA ECHO

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(Invited Communication)

Abstract—Extraction of wave-height directional spectral information from high-frequency (HF) radar sea echo requires the use of hydrodynamic and electromagnetic second-order coupling coefficients obtained from a perturbational expansion of the nonlinear boundary conditions at the ocean surface. To present, the hydrodynamic coupling coefficient derived for deep water has been given. Since most coastal HF radar observations are made in water shallow compared with the dominant ocean wavelength, that solution has proven inadequate for those applications. This paper derives the more general expression for water of arbitrary depth, and demonstrates its validity against measured data. The hydrodynamic contribution increases in importance as waves of constant energy move into shallow water. The use of these results for interpretation of both narrow-beam and CODAR data is discussed.

I. INTRODUCTION

THE potential of high-frequency (HF) radars for the measurement of sea state has been recognized since Crombie [1] experimentally observed the unique Doppler spectral signature of the sea echo. Barrick [2]–[4] derived theoretical expressions that explained these observed features quantitatively in terms of the ocean wave-height directional spectrum. To first order, the two dominant peaks in the spectrum are produced by Bragg scatter from the wave spectral components half the radar wavelength, traveling toward and away from the radar. At upper HF, these first-order peaks originate from short surface gravity waves (5–15 m long, with periods between 1.8 and 3.0 s); such waves are not the essence of “sea state,” where observers (for design, safety, and operational purposes) are generally concerned with waves having periods greater than 5 s. While it is conceptually possible to lower frequency in order to observe the wave-height directional spectrum in its significant region, this is quite impractical due to heavy radio spectral usage and severe hardware constraints (e.g., huge antenna sizes, the requirement of sweeping frequency over decades of bandwidth, absolute system gain and path loss knowledge at each frequency, etc.). Consequently, this dominant first-order echo has not proven useful for practical sea-state monitoring, although it is used successfully for mapping of surface currents with such radars [5]–[9].

Surrounding the first-order peaks in the echo spectrum are higher order peaks that are well above the noise; theoretical expressions for the nonlinear electromagnetic and hydrodynamic boundary conditions at the ocean surface [3], [4] based on their second-order perturbational solutions explain this echo structure. These integral solutions for the echo spectrum contain the wave-height directional spectrum. The kernel of the integrand is the “coupling coefficient,” which is the sum of the electromagnetic and hydrodynamic perturbational solutions to the surface boundary conditions. Both terms are presented by Barrick [3], [4]. The hydrodynamic coefficient for deep water gravity waves with wave vectors \( k, k' \) is derived in Weber and Barrick [10], [11]. The method used is identical to that pioneered by Hasselmann [12] and others for study of energy transfer due to nonlinear wave–wave interactions. These expressions for second-order sea scatter have been used successfully by Lipa to invert the HF radar return to obtain the ocean wave-height directional spectrum for both narrow-beamed antenna configurations [13]–[16], as well as echo from the more compact broad-beamed systems known as CODAR’s [17]–[20]. Because the echo spectral resolution must be high in order to extract detailed wave spectral information, success can be consistently assured only for radars operating in a “ground-wave” mode (i.e., propagation near the surface in contrast to “skywave,” or reflection from the ionosphere), where propagation path temporal variations do not distort the signal.

Most coastal ground-wave radar systems observe wave scatter in shallow water over the continental shelf, i.e., water whose depth is less than the dominant ocean wavelength to be extracted. Consequently, the methods discussed above will not work for this typical coastal situation. Although the electromagnetic coupling coefficient remains the same, the hydrodynamic does not, because the bottom boundary now becomes important in the perturbational solutions. The above analyses of second-order HF Doppler spectra show that the hydrodynamic contribution dominates for deep water (the electromag-
netic term cannot be neglected, however, because it yields directional information); energy-transfer studies [21] further indicate that nonlinear wave-wave contributions increase in shallow water. Consequently, having the correct hydrodynamic coupling coefficient is critical for extracting sea state from shallow-water measurements. We derive this coefficient in the next section. In Section III we show HF radar measurements in shallow water that confirm the validity of the expression. The last section then discusses the general problem of inverting the HF echo spectrum observed over a varying depth ocean, as is the case for broad-beam CODAR’s.

II. DERIVATION

The expression for the averaged normalized HF radar backscatter spectrum for vertically polarized second-order sea echo as a function of radian Doppler shift \( \omega^* \) from the carrier frequency (radar cross section per unit mean surface area per radians per second) is [3]

\[
\sigma^{(2)}(2k_0, \omega^*) = 2^6 \pi k_0^4 \sum_{m, m^* = \pm 1} \int \int \left| \Gamma_k \right|^2 \delta(mk) \delta(mk') \eta(m - m^* - m^* \omega') dp dq \]  

where the integration variables \( p \) and \( q \) are spatial wavenumbers with \( p \) aligned with the direction of the radar wave vector \( k_0 \), defined by the second-order Bragg constraint \( -2k_0 = k + k' \) as \( k = -(k_0, p, q) \) and \( k' = -(k_0, p, q) \). The factor \( \delta(y) \) is the Dirac-delta function of argument \( x \). The total coupling coefficient \( \Gamma_T = \Gamma_H + \Gamma_{EM} \) includes hydrodynamic (H) and electromagnetic (EM) contributions obtained from perturbation theory to second order. The wave vectors \( k \) and \( k' \) are understood to represent the two sets of waves interacting at second order in the radar scattering cell, i.e., at whatever that water depth \( d \) might be. The electromagnetic coupling coefficient, in terms of \( k, k' \), given in [3] and [4], remains the same versus water depth; the hydrodynamic does not, and will be derived here for arbitrary depth \( d \). The first-order radian temporal frequencies \( \omega \) and \( \omega' \) will also be related to wave vectors \( k, k' \) for water depth subsequently.

In a manner similar to [10], we express the rough water surface height at horizontal position \( r = (x, y) \) and time \( t \) as a spatial-temporal Fourier series, where height coefficients \( H(k, \omega) \) are indexed over integer multiples of arbitrary fundamental spatial and temporal frequencies \( K_x, K_y, W \) as \( k = (mk_x, nk_y) \) and \( \omega = \omega W \):

\[
\eta(r, t) = \sum_{k, \omega} H(k, \omega) \exp \{i(k \cdot r - \omega t)\} 
\]

where the Fourier coefficient for the surface height can be expressed in terms of its first- and second-order contributions as \( H(k, \omega) = H^{(1)}(k, \omega) + H^{(2)}(k, \omega) + \cdots \). The second-order coefficient will be shown to have the form

\[
H^{(2)}(k, \omega) = \sum_{k, \omega} \sum_{k, \omega'} \Gamma_H(k, \omega, k', \omega') H^{(1)}(k, \omega) H^{(1)}(k', \omega') \delta_{k, k'} \delta_{\omega, \omega'} 
\]

where the coupling coefficient \( \Gamma_H \) is the same as that required in (1). The factors \( \delta \) are Kronecker deltas, being unity when subscripts equal superscripts and zero otherwise. In addition, we use the fact that water velocities \( \nu \) in the ocean can be expressed in terms of a velocity potential \( \Phi \) that satisfies Laplace’s equation because on wave scales the water is incompressible and irrotational, i.e.,

\[
\nabla \cdot \nabla \phi(r, z, t) = 0 
\]

having a general solution expressible in a form similar to (2) as

\[
\phi(r, z, t) = \sum_{k, \omega} \left[ \Phi_\omega(k, \omega) e^{i\omega t} + \Phi_{-\omega}(k, \omega) e^{-i\omega t} \right] 
\]

where \( z \) is vertical distance (measured upward). For infinitely deep water (\( z \to -\infty \)), the constraint of a finite solution demands that the second term be zero; for finite depth, both \( \Phi_\omega \) and \( \Phi_{-\omega} \) are unknown coefficients to be determined to both first and second order (i.e., \( \Phi_\omega = \Phi_\omega^{(1)} + \Phi_\omega^{(2)} + \cdots \)), as well as \( H^{(1)} \), in terms of the first-order wave-height coefficient \( H^{(1)} \).

The boundary conditions include first the requirement that velocity normal to the bottom be zero, i.e.,

\[
u_z|_{z = -d} = \partial \phi / \partial z |_{z = -d} = 0.
\]

This allows us immediately to express \( \Phi_\omega \) and \( \Phi_{-\omega} \) in terms of a single \( \Phi \) (to all orders) so that (5) becomes

\[
\phi(r, z, t) = \sum_{k, \omega} \Phi(k, \omega) [e^{i(k \cdot z + \omega t)} - e^{-i(k \cdot z + \omega t)}] 
\]

\[
\cdot \exp \{i(k \cdot r - \omega t)\}. 
\]

The Navier–Stokes equation (coming from conservation of momentum) at the surface is

\[
[\partial \phi / \partial t + (1/2) \nabla \phi \cdot \nabla \phi]_{z = \eta} = -g \eta 
\]

where \( g \) is the acceleration of gravity, and the pressure above the free surface is zero. The second boundary condition is the kinematic constraint that a fluid particle at the surface remain on the surface, i.e.,

\[
[\partial \phi / \partial z]_{z = \eta} = \partial \eta / \partial t + \nabla \eta \cdot [\nabla \phi]_{z = \eta}, 
\]

At this point we substitute (2) and (7) into the boundary conditions (8) and (9). We employ the perturbational ordering of \( H \) as \( H^{(1)} + H^{(2)} \), and \( \Phi \) as \( \Phi^{(1)} + \Phi^{(2)} \). In addition to this ordering, we take the quantities \( \kappa_\eta \) and \( \kappa_\eta' \) to be ordering parameters, i.e., small compared to unity; for a single periodic wave train, they are waveslopes that have been used in classic Stokes analyses of higher order wave profiles. Finally, we expand all exponentials involving the above parameters into their series form, and retain terms through second order to
obtain
\[ \sum_{k, \omega} -2i\omega[\Phi^{(1)}(k, \omega) + \Phi^{(2)}(k, \omega)] \left[ \cosh (kd) + \sinh (kd) \sum_{k', \omega'} H^{(1)}(k', \omega') \cdot \exp [i(k' \cdot r - \omega' t)] \right] \]
\[ \cdot \exp [i(k \cdot r - \omega t)] + \frac{1}{2} \sum_{k, \omega} \sum_{k', \omega'} [4kk' \sinh (kd) \cdot \sinh (k'd) - 4k \cdot k' \cosh (kd) \cdot \cosh (k'd)] \]
\[ \cdot \exp [i(k \cdot r - \omega t)] + \frac{1}{2} \sum_{k, \omega} \sum_{k', \omega'} [4kk' \sinh (kd) \cdot \cosh (kd) \cdot \cosh (k'd)] \]
\[ \cdot \exp [i(k \cdot r - \omega t)] - i(\omega + \omega')t] = \sum_{k, \omega} \sum_{k', \omega'} H^{(1)}(k, \omega) H^{(1)}(k', \omega') \delta_{k+k'} \delta_{\omega+\omega'} = 0 \]
(10)
and
\[ \sum_{k, \omega} 2k[\Phi^{(1)}(k, \omega) + \Phi^{(2)}(k, \omega)] \left[ \sinh (kd) + \cosh (kd) k \right] \]
\[ \cdot \sum_{k', \omega'} H^{(1)}(k', \omega') \cdot \exp [i(k' \cdot r - \omega' t)] \]
\[ \cdot \exp [i(k \cdot r - \omega t)] + \sum_{k, \omega} \sum_{k', \omega'} 2k \cdot k' \Phi^{(1)}(k, \omega) \]
\[ \cdot H^{(1)}(k', \omega') \cosh (kd) \exp [i(k + k') \cdot r] \]
\[ - i(\omega + \omega')t] = \sum_{k, \omega} \sum_{k', \omega'} \{ f(k, k') + f(k', k) / 2 \} \delta_{k+k'} \delta_{\omega+\omega'} \] \[ \sum_{k, \omega} \sum_{k'} \{ f(k, k') + f(k', k) / 2 \} \delta_{k+k'} \delta_{\omega+\omega'} = 0. \]
(11)

Since these are Fourier series in space and time \((r, t)\), we eliminate these variables to obtain equations in the Fourier coefficients. This is done by multiplying by \(\exp [i(k \cdot r - \omega t)]\) and integrating over the fundamental spatial and temporal periods of the series. When we separate out and equate the first-order terms, we arrive at the following equations

\[ -2i\omega \Phi^{(1)}(k, \omega) \cosh (kd) = -gH^{(1)}(k, \omega) \]
\[ 2k\Phi^{(2)}(k, \omega) \sinh (kd) = -i\omega H^{(1)}(k, \omega). \]
(12)

Dividing the second by the first gives the shallow-water dispersion relation
\[ \omega^2 = g\omega \tanh (kd). \]
(13)

We now follow the same procedure for the second-order terms; rather than eliminating one of the sets of spatial-temporal wavenumbers, we carry both \((k, \omega, k', \omega')\) along, expressing their relationship with \(k'', \omega''\) through Kronecker deltas. Also, we use (12) and (13) to eliminate the first-order velocity potential coefficient, to obtain the following two equations involving the second-order wave height and velocity potential coefficients:

\[ -2i\omega \Phi^{(1)}(k'', \omega'') \cosh (k''d) + gH^{(2)}(k'', \omega) + \sum_{k, \omega} \sum_{k', \omega'} \]
\[ \cdot \left[ -\omega^2 - \frac{1}{2} \omega \omega' (1 - k \cdot k' \coth (kd) \coth (k'd)) \right] \]
\[ \cdot H^{(1)}(k, \omega) H^{(1)}(k', \omega') \delta_{k+k'} \delta_{\omega+\omega'} \]
\[ = 0 \]
(14)

and

\[ 2k''\Phi^{(2)}(k'', \omega'') \sinh (k''d) - i\omega'' H^{(1)}(k'', \omega'') \]
\[ + i \sum_{k, \omega} \sum_{k', \omega'} \left[ -\frac{gk''}{\omega} - \frac{gk \cdot k'}{\omega} \right] \]
\[ \cdot H^{(1)}(k, \omega) H^{(1)}(k', \omega') \delta_{k+k'} \delta_{\omega+\omega'} = 0. \]
(15)

Second-order velocity potential \(\Phi^{(2)}(k'', \omega'')\) is now eliminated from these equations to obtain a single equation for the second-order wave-height coefficient \(H^{(2)}(k'', \omega'')\). Next, we perform mathematical manipulations to make the summations symmetric in \(k, \omega, k', \omega'\). This is done by redefining summation indices; e.g., if \(f(k, k')\) is a term in the double series that is not symmetric in \(k, k'\), we can make the series symmetric by using the fact that

\[ \sum_{k} \sum_{k'} f(k, k') \delta_{k+k'} = \sum_{k} \sum_{k'} \{ f(k, k') + f(k', k) / 2 \} \delta_{k+k'}. \]

Finally, we add and subtract the same terms in order to cast the result for \(H^{(2)}\) in the form of (3), where the coupling coefficient resembles that derived in [10] for deep water, obtaining

\[ T_{kk'}(k, \omega, k', \omega') \]
\[ = \frac{1}{2} \left\{ k \tanh (kd) k' \tanh (k'd) \right. \]
\[ \left. + g \frac{kk' \tanh (kd) \tanh (k'd) - k \cdot k'}{\omega \omega'} \right. \]
\[ \left. + g \frac{\left[ gk'' \tanh (k''d) + \omega''^2 \right]}{\left[ gk'' \tanh (k''d) - \omega''^2 \right]} \right\} \]
\[ \left. \omega'' [\omega \cosh (kd) + \omega''^2 \cosh (k'd)] / g \right\} \]
\[ gk'' \tanh (k''d) - \omega''^2 \]
\[ = gk \tanh (kd) \]
(16)

where we understand the first-order dispersion relations between \(\omega, k\) and \(\omega', k'\), i.e.,

\[ \omega^2 = gk \tanh (kd) \]
and
\[ \omega'^2 = gk' \tanh (k'd) \]
but no such relationship exists between second-order wavenumbers \(k'', \omega''\). The latter obey the constraints given by
the Kronecker deltas in (3), i.e.,

\[ k'' = k + k' \]

and

\[ \omega'' = \omega + \omega'. \]

It is obvious that this coupling coefficient reduces to that for deep water given in [3], and [10]:

\[
\Gamma_W(k, \omega, k', \omega') \xrightarrow{d \to \infty} \frac{1}{2} \left\{ k' + k + g \frac{gk' - k \cdot k'}{\text{Re} \left( \frac{gk'' + \omega''^2}{gk'' - \omega''^2} \right)} \right\}. \tag{17}
\]

III. EXPERIMENTAL VALIDATION

To test the formula for the shallow-water coupling coefficient (16) we use measurements made by a CODAR system at Pescadero in 1978. The experimental situation is illustrated in Fig. 1. The CODAR receiving antenna consists of two crossed loops and a monopole, signals from which are combined to extract directional information. Radar measurements were obtained in 1.2-km range cells to distances of 20 km. On January 19, the dominant sea waves consisted of 16-s swell from the west, which was observed both by radar and a Scripps pitch-and-roll buoy [18]. Such long ocean waves are effectively in shallow water as they move through the CODAR coverage area. Fig. 2 shows the narrow-beam coupling coefficient (16) plotted as a function of water depth for 16-s waves moving directly along the radar beam for a frequency of 25.4 MHz, normalized to that for deep water (17).

We now describe an experimental verification of the shallow-water results for the broad-beamed CODAR measurements, based on data from the omnidirectional monopole antenna. Equation (1) shows that the narrow-beam second-order radar cross section in shallow water is expressed as a two-dimensional integral, whose integrand contains the coupling coefficient as a multiplicative factor. Due to the growing value of the coupling coefficient with decreasing water depth, illustrated theoretically in Fig. 2, the second-order portion of the radar Doppler spectrum will increase in magnitude as the waves move into shallower water, i.e., as radar range decreases. In contrast, the first-order spectral magnitude, which is due to scatter from short ocean waves (i.e., 2-s period for 25.4-MHz radar frequency), remains unchanged for water depth greater than 2 m. The total sea-echo spectrum measured by the monopole is the integral over azimuth angle of the narrow-beam radar cross section, and as the depth decreases over the close-in range cells (Fig. 1), the second-order echo spectrum increases relative to the first. This was observed at Pescadero and is shown in Fig. 3 for the monopole signals. To perform a quantitative test, we form two estimates—experimental and theoretical—of the ratio of the energy in the dominant second-order peak to that in the neighboring first-order peak at each range. The first is obtained directly from the measured radar spectrum; the normalization removes unknown path losses and system effects. The second is obtained from the theoretical formulation, substituting the wave-height directional spectrum measured by the buoy, and integrating over azimuth angle. Hence, the coupling coefficient and wave-directional spectrum are both evaluated in the angular integration as a function of the actual depth within each range cell. (More detail of this process is given in the next section.) Results are shown in Fig. 4. The agreement between theory and experiment over a fairly large magnitude span therefore demonstrates the validity of the shallow-water coupling coefficient (16) derived here, as well as the methods used to calculate the shallow-water wave spectrum based on its deep-water value.
sensitive to depth, being significant in deep water, but increasing further in its importance for the same wave spectrum as it moves into shallow water. This was derived theoretically here and confirmed by measurements; it is also consistent with the increase of energy transfer for shallow water theoretically suggested in the work by Herterich and Hasselmann [21] due to nonlinear wave-wave interactions. Improper use of the deep-water coefficient in the inversion of HF radar echo in shallow water will result in underprediction of wave energy. One cannot employ this or any known coefficient in extremely shallow water, where other physical processes ignored here become dominant; e.g., shoaling in the surf zone, bottom percolation, etc. A good rule of thumb that has been followed [22] is that the methods and the shallow-water dispersion relation can be used in water whose depth exceeds $l/20$ the dominant wavelength on deep water.

The interpretation of narrow-beam radar results [20] assumes a radar cell whose angular width is small (e.g., $<10^\circ$). It also assumes that the cell spatial dimensions are sufficiently small in terms of horizontal depth variations that the wavefield within the cell does not change statistically from point to point. In this event, one can use the deep-water methods for inversion of second-order Doppler spectra with confidence in shallow water by a) substituting the shallow-water hydrodynamic coupling coefficient derived here for the deep-water value; and b) employing the shallow-water dispersion relation (13) everywhere it occurs, e.g., in the electromagnetic coupling coefficient, in the Dirac-delta function constraint, etc.

Interpretation of CODAR data in coastal situations for wave extraction is generally more complex [20]. A radar cell for a given antenna element has an angular span that can be as great as $180^\circ$, i.e., a semicircle bounded by the coastline. The water depth in such a cell will typically range from very shallow at its edges, to deep straight out from shore. Hence, the radar measurements are made over varying depth; the formulation for the second-order echo must contain depth, i.e., be a function of position. With the assumption of onshore wave fields that would have been homogeneous in the CODAR coverage area (e.g., $<20$-km radius) had the water been everywhere deep, the general software approach we have used successfully a) stores the depth in the coverage area versus position; b) allows the coupling coefficient to change with position according to water depth; c) allows the dispersion relation to vary in this manner also; and d) uses Snell's law for wave refraction to relate angular changes in wave energy to depth (i.e., ray tracing), which is valid for the situations to which the methods employed in this paper apply. Hence, we express the shallow-water spectrum in the integrand of (1) at each point within the measurement area in terms of the deep-water spectrum; the latter is then determined through inversion of the integral equation. The wave-height directional spectrum at any desired point in shallow water can be determined by Snell's law, and the dispersion relation from the deep-water spectrum. The software inversion for each shallow-water coastal CODAR location is therefore site specific, in the sense that the required depth-contour arrays pertain to that site.

**IV. APPLICATION AND DISCUSSION**

The coupling coefficient required for interpretation and inversion of second-order HF Doppler radar spectra of sea echo is obtained from the nonlinear boundary conditions at the ocean surface and bottom. The hydrodynamic contribution is
REFERENCES


Donald E. Barrick (M'62), for photograph and biography please see this issue, p. 146.

Belinda J. Lipa, for photograph and biography please see this issue, p. 245.