Geometrical Dilution of Statistical Accuracy (GDOSA) in Multi-Static HF Radar Networks

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• **Introduction and Rationale:** We in the field of HF radars are now well familiar with the term GDOP: geometrical dilution of precision. We think of this as the baseline instability problem along the line joining two backscatter current-mapping HF radars. In the region near this "baseline', one cannot produce accurate total vectors because the two sets of radial velocities lie nearly parallel to each other.

Chapman et al. (*J. Geophys. Res.,* vol. 102, pp. 18,737-18,748, 1997) treated GDOP for a pair of backscatter radars, deriving the relations for this case. While it was a needed contribution, I maintain that it is not a complete nor totally accurate picture even for two backscatter radars, let alone multiple radars that view the same point: some backscatter and some bistatic. Here is a summary of what I include below, along with a comparison of how it relates to the GDOP of Chapman et al.

• We show how to calculate the uncertainty in 'u' and 'v', as well as the uncertainty in 'w', the total velocity. Chapman et al. (1997) dealt with GDOP of u and v for two backscatter radars. When one gets too close to the baseline, the total vector calculated usually goes crazy, sometimes getting huge, always in directions perpendicular to the baseline, so this discussion is certainly needed. However, the uncertainty in 'u' and/or 'v' have totally different transformations: one of them actually gets better near the baseline. This is important for data assimilation into models, because one could assimilate the better component and omit the worse.

• We extend this to mixes of arbitrary numbers of bistatic (with elliptical geometries) and backscatter radars (with radial geometries), where scalar radar components from several radars may be available at one point on the sea.

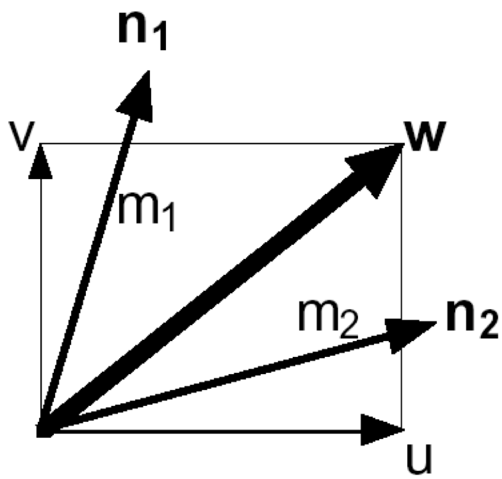
• We include the effects of the number of statistical samples available for the calculation, which increases close to the radar where fixed angle grids provide more measurements per unit area. The peculiar grids of bistatic radars are also incorporated. None of this has been considered before with standard GDOP analyses, but is equally important to final u/v and total vector accuracy. This also allows one to study how increasing the spatial resolution of one or more radars may improve map accuracy.

• We consider how these concepts may be readily extended to calculate spatial covariance matrices needed in the Kalman filters for assimilation into numerical models.

• Finally, we show subsequently (for other purposes) how to calculate the transformation of resolution from the radar-peculiar radial and elliptical coordinates into the Cartesian x/y grid on which users desire to plot total vectors.

- **System of Equations and Deriving u/v Uncertainty for Two Backscatter Radars:**

For simplicity and to get our bearings, start with two radar measurements of vector components m_1 and m_2 at a desired grid point. These could be radials or ellipticals. From these, the actual total velocity $\mathbf{w} = (u,v)$ is to be derived. Unit vectors or normals along the radar-measured velocity components are $\mathbf{n_1}$ and $\mathbf{n_2}$. The components of the unit normals are defined with respect to the x and y axes (whose directions are arbitrary in general, but x here is taken to lie along the baseline for two coastal backscatter radars). These are all shown below.



The quantities that are known are: m_1, m_2, $\mathbf{n_1}$, and $\mathbf{n_2}$. The quantity that is unknown and is sought is the total velocity , or alternately, its East (x) and North (y) components, u and v.

The set of two linear equations for these relations are:

$$\begin{aligned} m_1 &= n_{1x}u + n_{1y}v \\ m_2 &= n_{2x}u + n_{2y}v \end{aligned} \qquad (1)$$

whose solutions are:

$$\begin{aligned} u &= \frac{n_{2y}m_1 - n_{1y}m_2}{n_{1x}n_{2y} - n_{2x}n_{1y}} \\ v &= \frac{n_{1x}m_2 - n_{2x}m_1}{n_{1x}n_{2y} - n_{2x}n_{1y}} \end{aligned} \quad . \qquad (2)$$

The denominator in both of these is the sine of the angle between $\mathbf{n_1}$, and $\mathbf{n_2}$, which goes to zero when the vectors become parallel or anti-parallel. This is related to the GDOP as defined by Chapman, and itself can be considered a gross measure of the stability of the ***total*** velocity, $\mathbf{w}$. More meaningful measures can be assigned to the u and v components themselves. We examine these concepts next.

Again, for ultimate simplicity at this point, assign the same uncertainty or random error to the two measured quantities, m_1 and m_2. *After all, without some measure of error, GDOP is meaningless, because one can estimate* u and v *perfectly right up to the parallel condition if there is no measurement error.* (Calculations on GPS display readouts also show GDOP. The uncertainty displayed there comes from the measured Signal-to-Noise ratio.) To keep the problem simple with minimal assumptions and parameters, take the errors in measured radial or elliptical velocities to be zero-mean about the average value of the measurements, assume they are uncorrelated with each other, and that their standard deviations (square root of their variances) are equal to each other: σ. We will remove all of these restrictions later.

With the assumptions of zero-mean uncorrelated radial/elliptical velocity errors with equal variances, the variances in u and v can be written almost from inspection. They are:

$$\sigma_u^2 = \frac{n_{2y}^2 + n_{1y}^2}{\left|n_{1x}n_{2y} - n_{2x}n_{1y}\right|^2}\sigma^2$$

$$\sigma_v^2 = \frac{n_{2x}^2 + n_{1x}^2}{\left|n_{1x}n_{2y} - n_{2x}n_{1y}\right|^2}\sigma^2$$

and therefore for the total velocity, (3)

$$\sigma_w^2 = \sigma_u^2 + \sigma_v^2 = \frac{2\sigma^2}{\left|n_{1x}n_{2y} - n_{2x}n_{1y}\right|^2}$$

Note that the denominator, $\left|n_{1x}n_{2y} - n_{2x}n_{1y}\right|$, is identically $\sin(\varphi)$, where φ is the angle between the lines from the radar sites to the observation point: when this is 0° or 180°, GDOP becomes infinite. The third equation above may be the reasoning that Chapman et al. used to justify calling GDOP simply the absolute value of the sine of the angle between the measurement vectors. The other two, I maintain, are meaningful because: why throw away the entire vector when either the u or the v might be very accurate, and could provide a valuable input to model assimilation? Simply set up GDOP acceptability threshholds on each component (or weight them in a maximum-likelihood fashion by these uncertainties).

- **Generalization of Equation System to N Multi-Static Radars:** One can almost immediately write the extension to the first set of equations above to allow for N observations at a point, and do all of our required calculations with matrix algebra.

$$m_1 = n_{1x}u + n_{1y}v$$

$$m_2 = n_{2x}u + n_{2y}v$$

$$m_3 = n_{3x}u + n_{3y}v \qquad (4a)$$

$$m_N = n_{Nx}u + n_{Ny}v$$

or

$$[M] = [N][W] \qquad (4b)$$

where $[M]$ is the $N \times 1$ measured data vector; $[N]$ is the $N \times 2$ known geometry vector of the direction cosines along the measured scalar component; and $[W] = (u, v)$ is the 2×1 unknown total velocity vector resolved into its East/North components.

Note first that this is an overdetermined system of equations (more equations than unknowns), and so it cannot be solved exactly. Its optimal solution is based on a least-squares minimization of the mean-squared differences between data (left side) and model (right side). This least-squares solution multiplies both sides on the left by the transpose of $[N]$ to get a square 2×2 system on the right side, and then inverts this to get the desired solution:

$$[W] = \left[[N]^T[N]\right]^{-1}[N]^T[M] \qquad (5)$$

where as stated before, $[N]^T[N]$ is a square, symmetric 2×2 system, as is its inverse, $\left[[N]^T[N]\right]^{-1}$. This is therefore the method used to determine the total velocity when more than two radars see a given point on the sea.

Now we use the above least-squares matrix methods to obtain the covariance matrix of the desired total vector in terms of the covariance matrix of the measured data. First, write the matrices of the observed and desired current velocities in terms of a mean and statistical fluctuating component. By this definition, the mean of the fluctuating component is zero. Thus we have:

$$[M] \equiv [\overline{M}] + [m] \qquad (6)$$

and

$$[W] \equiv [\overline{W}] + [w]. \qquad (7)$$

We substitute these into Eq. (4a) above. Note that the solution for the mean is the same as (4b) and (5), i.e.,

$$\left[M\right] = \left[N\right]\left[W\right] \qquad (8)$$

and hence

$$\left[\overline{W}\right] = \left[\left[N\right]^T\left[N\right]\right]^{-1}\left[N\right]^T\left[\overline{M}\right]. \qquad (9)$$

The means can be eliminated from the result after substitution. Then we post-multiply by the transpose to get:

$$\left[m\right]\left[m\right]^T = \left[N\right]\left[w\right]\left[w\right]^T\left[N\right]^T \qquad (10)$$

We average this equation and use the following definitions of covariance matrices.

$$\left[C_m\right] \equiv \left\langle\left[m\right]\left[m\right]^T\right\rangle \text{ and } \left[C_w\right] \equiv \left\langle\left[w\right]\left[w\right]^T\right\rangle \qquad (11)$$

to get

$$\left[C_m\right] = \left[N\right]\left[C_w\right]\left[N\right]^T \qquad (12)$$

Note that the unknown $\left[C_m\right]$ is a $N \times N$ matrix while the desired $\left[C_w\right]$ is a 2×2 matrix. Again, we have an overdetermined system, whose least-squares solution results in pre- and post-multiplying by transposes to make the matrices surrounding $\left[C_w\right]$ square, then pre- and post-multiplying by the inverses to get:

$$\left[C_w\right] = \left[\left[N\right]\left[N\right]^T\right]^{-1}\left[N\right]^T\left[C_m\right]\left[N\right]\left[\left[N\right]\left[N\right]^T\right]^{-1} \qquad (13)$$

(Note that the transposes of the inverses are the same as the original inverses because of their symmetry.) ***The above equation is the desired result for the uncertainties in the total velocities.*** To convince yourself, reduce this to the 2×2.

- **The Input Data Covariance Matrix, $\left[C_m\right]$ Based On Number of Samples:** In the simple 2×2 backscatter case leading up to Eqs. (3), we took the data covariance matrix for the radials to be diagonal with equal variances for both sets of radials, σ. We now want to generalize.

First of all, we argue that this data covariance matrix is still diagonal. This is justified because the radial (or elliptical) velocity measured by one radar is statistically independent from that of the other at the same point. They view different Bragg waves from different directions. Because such waves are not statistically correlated, neither are their signals. Hence the off-diagonal covariance elements are zero.

However, the variance of the radial or elliptical velocity depends inversely on the number of samples used in the data averaging process. All other things being equal, this depends inversely on the area of the radar cell with respect to some fixed area measure. For example, suppose that a point within a 2 x 2 km Cartesian grid at which the total vector is desired is very close to a backscatter radar. There may be many radial velocity vectors to be averaged in a $\Delta A = 2 \times 2 \text{ km}^2$ Cartesian because of the polar bearing grid. Thus the number of points in the average is proportional to $\Delta A/(R\,\Delta R\,\Delta\vartheta)$, where the denominator is the polar cell area at that Cartesian grid point. The latter, of course, is calculable at each point of the map grid. Thus, the input data matrix for a backscatter radars will vary with position: the further from the radar, the worse is the radial velocity variance because of fewer samples.

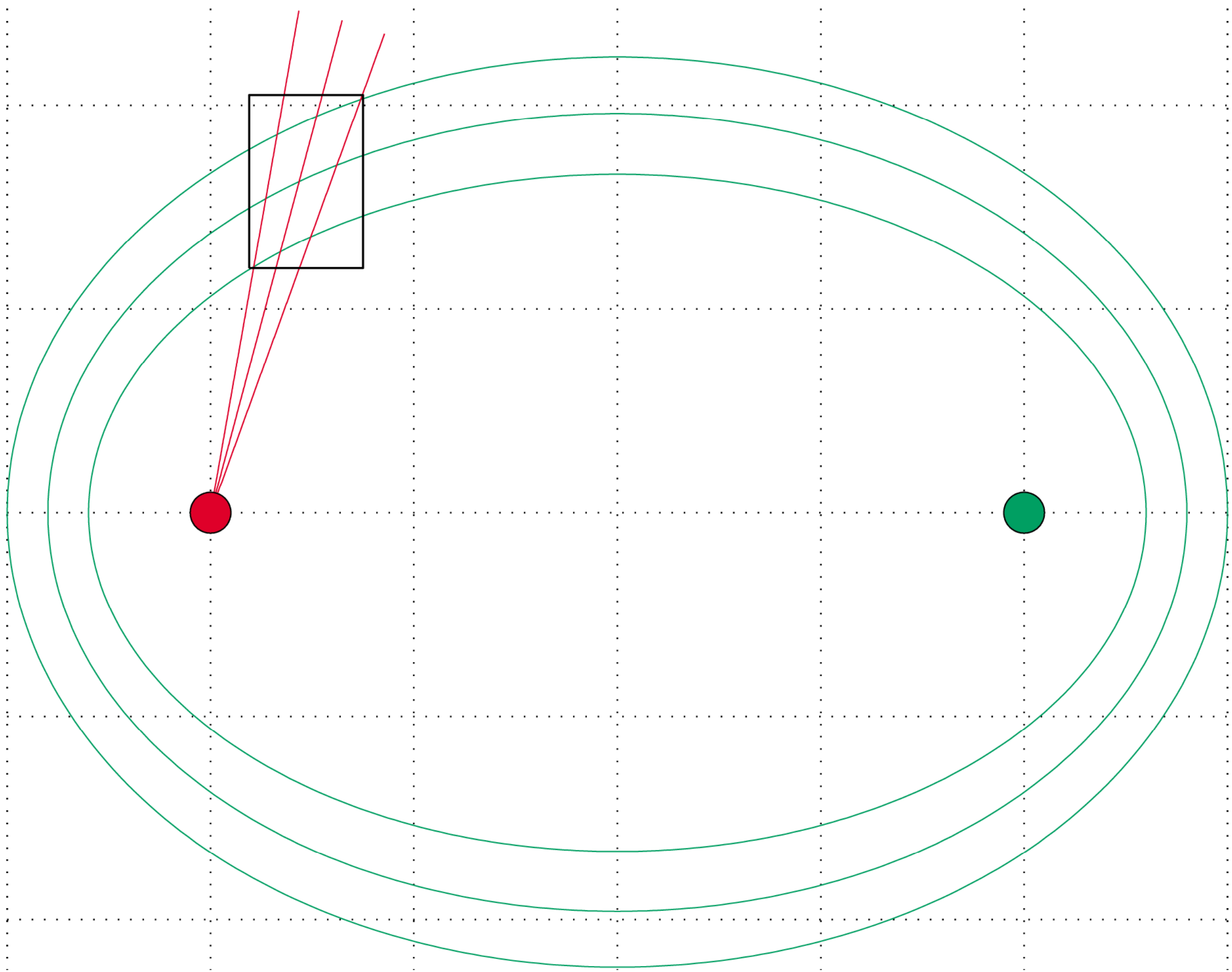
We can therefore write the input data covariance matrix as proportional to:

$$\left[C_m\right] \sim \frac{1}{\Delta A_C}\begin{bmatrix} \Delta A_1 & 0 & 0 & 0 \\ 0 & \Delta A_2 & 0 & 0 \\ 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \Delta A_N \end{bmatrix} \qquad (14)$$

where ΔA_C is the reference area of the final Cartesian grid cell, and ΔA_n is the area of the cell from the n-th radar at the desired Cartesian grid point. For a backscatter radar, we stated above that this is: $(R\,\Delta R\,\Delta\vartheta)$.

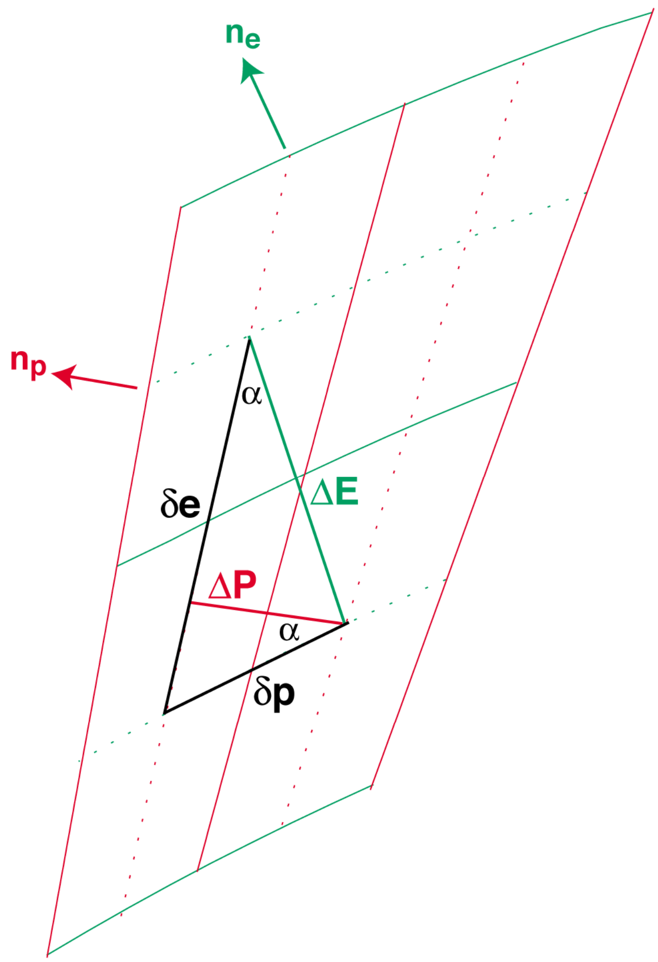
- **The Radar Cell Size for Bistatic Geometries:** For a bistatic radar observation at the Cartesian grid point, one needs to find the corresponding area cell size. We illustrate how to calculate this for radar cells defined for the bistatic geometry below, where the receiver is on the left (red circle) and the transmitter on the right (green circle). The angular extent of the cell is defined by bearing spokes from the receiver on the left, because that is where the bearing-determining antenna is located. The cells are determined by the intersections of these receiver bearing spokes with the elliptical time delay cells.

The rectangular box in the first figure is enlarged in the second figure to show the differential cells about an arbitrary point in the bistatic system. In this region, the spatial distance perpendicular to the elliptical cells is related to the time-delay increment, and is calculable; we denote it by ΔE here. The unit normal perpendicular to the ellipses at this point is denoted $\mathbf{n_e}$. Both the cell spacing, ΔE, and the unit normal, $\mathbf{n_e}$, along which the measured bistatic scalar velocity lies (as needed for Eqs. (4)) are calculated at the desired Cartesian grid point from the standard bistatic/elliptical coordinate relations.



Likewise, the spatial increment perpendicular to the increasing direction of receiver bearing angle is $\Delta P = R_R \Delta\vartheta$, where R_R is the distance from the receiver to the point under consideration, and $\Delta\vartheta$ is the bearing-angle grid increment or resolution (5° for most SeaSonde operations). The unit normal perpendicular to increasing bearing direction is $\mathbf{n_p}$.

The cell walls approximate a parallelogram to first order. The area within any parallelogram is then given by the product of the perpendicular distances between the parallel walls, i.e., $\Delta A_e = \Delta E \, \Delta P$. This is the quantity to be used in the matrix of Eq. (14) when bistatic cells are needed. The quantity ΔE is calculated at the desired Cartesian grid point from the standard bistatic/elliptical coordinate relations, while the quantity $\Delta P = R_R \Delta\vartheta$.



• **Data Covariance Matrix at Cartesian Grid Points:** Using the above methods, we have determined through Eq. (13) the covariance matrix at the Cartesian grid points. At a single point, this is a simple 2 x 2 matrix. However, note that this matrix is not diagonal, even though the input data covariance matrix was diagonal. Thus, the East and North velocity components, u, v, in general are correlated with each other. This fact may be important for some data interpretation applications, like assimilation into numerical models.

In the case where only the total velocity variance -- the GDOSA -- is desired (not those of u and v individually), we can define this quantity by summing the two diagonal elements, as we did for the two-radar case in the last of Eqs. (3). That is:

$$\sigma_w^2 = \sigma_u^2 + \sigma_v^2 \qquad (15)$$

The above quantity σ_W from Eqs. (15), (13), and (14) above are what I have been plotting as color contour intensities with my MATLAB code and calling "Quality Factor" in examining placements of transmitters and receivers in multi-static radar geometries.

- **GDOSA with Measured Data:** The above treatment is not meant to be used with after-the-fact measured data. It is intended solely to estimate quality for proposed HF radar siting and geometries, in order to make decisions about numbers of radars and deployment locations.

With measured data processed properly at the radial level (for backscatter) or elliptical level (bistatic), the above methodology is already incorporated into CODAR's processing algorithms. Rather than making unsupportable assumptions, e.g., that all of the input scalar radial or elliptical measurements have the same uncertainty, the latter quantities are measured at each grid point for the radial/elliptical measurement. The methods by which this is done are described in the following reference, available as PDF from our website: [Lipa, B., "Uncertainties in SeaSonde Current Velocities", *Proc. of the IEEE/OES Seventh Working Conference on Current Measurement Technology*, Judith A. Rizoli, Ed., New York, March 2003, IEEE].

The equations for calculating total vectors that are weighted inversely by the uncertainties (i.e., maximum likelihood) are given in the above reference for multiple radar observations at the same point. Likewise, the uncertainties in u/v -- expressed through their covariance matrix -- is also given. Structurally, it is the same as that given in these notes, but based on the actual data. Furthermore, it inherently includes all of the geometry (GDOP/GDOSA) issues, without any artificial introduction of a "mask" or overlay, as we frequently hear is necessary. Thus, for example, near the baseline, one would expect to see the covariance-matrix component -- rotated to be perpendicular to that baseline -- will be large, reflecting the near-parallel geometry. There is no need to separately introduce such a quality factor, after the fact. This covariance matrix -- calculated in the manner described in the above reference -- is now outputted as standard procedure in CODAR's total vector files.